MATH2060 TUT01

Def. Let $I \subseteq \mathbb{R}$ be an interval, $f: I \to \mathbb{R}$ be a fcn, $c \in I$.

- We say that f is differentiable at $c \in I$ if
$$f'(c) := \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \quad \text{exists}$$
We call $f'(c)$ the derivative of f at c.
- We say that f is diff. on I if $f'(x)$ exists $\forall x \in I$.

Thm 6.2.4 (Mean Value Thm)

Suppose
- $f: [a,b] \to \mathbb{R}$ is cts $\quad (a<b)$
- $f'(x)$ exists $\forall x \in (a,b)$

Then $\exists\ c \in (a,b)$ s.t.
$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Thm 6.2.7 Let $f: I \to \mathbb{R}$ be diff. on an interval I. Then

a) f is increasing on I $\iff$ $f'(x) \geq 0 \ \forall x \in I$

b) f is decreasing on I $\iff$ $f'(x) \leq 0 \ \forall x \in I$

Example (§6.1 Ex 7)

Suppose that f is diff. at c and that $f(c) = 0$.
Show that $g(x) := |f(x)|$ is diff. at c iff $f'(c) = 0$.

Ans: $\forall x \neq c$,

$$\frac{g(x) - g(c)}{x - c} = \frac{|f(x)| - |f(c)|}{x - c} = \operatorname{sgn}(x-c) \left| \frac{f(x) - f(c)}{x - c} \right|$$

since $f(c) = 0$.

Here $\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$

Then $$\lim_{x \to c^+} \frac{g(x) - g(c)}{x - c} = \lim_{x \to c^+} \operatorname{sgn}(x-c) \left| \frac{f(x) - f(c)}{x - c} \right| = |f'(c)|$$

$$\lim_{x \to c^-} \frac{g(x) - g(c)}{x - c} = \lim_{x \to c^-} \operatorname{sgn}(x-c) \left| \frac{f(x) - f(c)}{x - c} \right| = -|f'(c)|$$

Since $\displaystyle\lim_{x \to c} \frac{g(x) - g(c)}{x - c}$ exists iff $\displaystyle\lim_{x \to c^+} \frac{g(x) - g(c)}{x - c} = \lim_{x \to c^-} \frac{g(x) - g(c)}{x - c}$,

so $g'(c)$ exists iff $|f'(c)| = -|f'(c)|$
iff $f'(c) = 0$.

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Example (§6.1 Ex 13)

If $f: \mathbb{R} \to \mathbb{R}$ is diff. at $c \in \mathbb{R}$, show that

$$f'(c) = \lim \left(n \{ f(c + 1/n) - f(c) \} \right)$$

However, show by example that the existence of the limit of this seq. does not imply the existence of $f'(c)$

Ans: Since f is diff. at c,

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

Consider the seq $\{h_n\}$, $h_n := \frac{1}{n}$,

we have $h_n \neq 0$ and $\lim(h_n) = 0$

By Sequential Criterion for Limits of fcns,

$$\lim_{n \to \infty} \frac{f(c+h_n) - f(c_n)}{h_n} = f'(c)$$

i.e. $f'(c) = \lim_{n \to \infty} n[f(c + \frac{1}{n}) - f(c)]$ //

For the counterexample, one may consider the Dirichlet fcn

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then $n[f(c+\frac{1}{n}) - f(c)] = \begin{cases} n(1-1) & \text{if } c \in \mathbb{Q} \\ n(0-0) & \text{if } c \notin \mathbb{Q} \end{cases}$

$= 0 \quad \forall n \in \mathbb{N}$.

However, $f'(c)$ DNE for any $c \in \mathbb{R}$ since f is discts everywhere //

Example ($\S$6.2 Ex 8)

Let $f:[a,b]\to\mathbb{R}$ be cts on $[a,b]$ and diff. on (a,b).
Show that if $\lim_{x\to a} f'(x) = A$,
then $f'(a)$ exists and equal A.

Ans: Idea: By MVT, $\dfrac{f(x)-f(a)}{x-a} = f'(c_x) \to A$

as $x \to a^+$

Let $\varepsilon > 0$.
Since $\lim_{x\to a} f'(x) = A$, $\exists\, \delta > 0$ s.t.
if $x \in (a,b)$ and $0 < |x-a| < \delta$,
we have $|f'(x) - A| < \varepsilon$

Fix $x \in (a,b)$ s.t. $0 < |x-a| < \delta$.
Apply Mean Value Thm to the interval $[a,x]$.
Then $\exists\, c_x \in (a,x)$ s.t.

$$\frac{f(x)-f(a)}{x-a} = f'(c_x)$$

Note $0 < c_x - a < x - a < \delta$.
Hence $\left|\dfrac{f(x)-f(a)}{x-a} - A\right| = |f'(c_x) - A| < \varepsilon$

Therefore $\displaystyle\lim_{\substack{x\to a\\ x\in[a,b]}} \frac{f(x)-f(a)}{x-a} = A$

Since a is the left end pt. of $[a,b]$, it means
$f'(a) = A$

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Example (§6.2 Ex 5)

Let $a > b > 0$ and let $n \in \mathbb{N}$ satisfy $n \geq 2$.
Prove that $a^{1/n} - b^{1/n} < (a-b)^{1/n}$

Ans: Divide $b^{1/n}$ on both sides, we have

$$\left(\frac{a}{b}\right)^{\frac{1}{n}} - 1 < \left(\frac{a}{b} - 1\right)^{\frac{1}{n}}$$

$$\left(\frac{a}{b}\right)^{\frac{1}{n}} - \left(\frac{a}{b} - 1\right)^{\frac{1}{n}} < 1.$$

This leads us to consider the fcn $f(x) := x^{\frac{1}{n}} - (x-1)^{\frac{1}{n}}$ for $x \geq 1$.

Let $f(x) := x^{\frac{1}{n}} - (x-1)^{\frac{1}{n}}$ for $x \geq 1$.

Then $f'(x) = \frac{1}{n} x^{\frac{1}{n}-1} - \frac{1}{n}(x-1)^{\frac{1}{n}-1}$ for $x > 1$. ($f'(1)$ DNE)

Moreover, for $x > 1$,

$$x > x - 1 > 0$$

$$\Rightarrow 0 < x^{\frac{1}{n}-1} < (x-1)^{\frac{1}{n}-1} \quad \text{since } \frac{1}{n} - 1 < 0$$

$$\Rightarrow f'(x) < 0$$

As f is cts on $[1,x]$ and diff. on $(1,x)$,
MVT implies that $\exists\, c_x \in (1,x)$ s.t.

$$\frac{f(x) - f(1)}{x - 1} = f'(c_x) \quad (< 0)$$

$$\Rightarrow \quad f(x) < f(1)$$

Hence $f(x) < 1 \quad \forall x > 1.$

Wish to apply Thm 6.2.7, but it requires diff. on whole interval, so prove directly by MVT instead.

Finally, $a > b > 0 \Rightarrow \frac{a}{b} > 1$

and so $f\left(\frac{a}{b}\right) < 1$,

i.e. $\left(\frac{a}{b}\right)^{\frac{1}{n}} - \left(\frac{a}{b} - 1\right)^{\frac{1}{n}} < 1$

$\Leftrightarrow \quad a^{\frac{1}{n}} - b^{\frac{1}{n}} < (a-b)^{\frac{1}{n}}$

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Example (§6.2 Ex 10)

Let $g: \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) := \begin{cases} x + 2x^2 \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 . \end{cases}$$

Show that $g'(0) = 1$, but in every neighbourhood of 0 $g'(x)$ takes on both +ve and -ve values.

Thus g is NOT monotonic in any neighbourhood of 0.

Ans: $x \neq 0$: By chain rule and product rule,

$$g'(x) = 1 + 4x\sin(1/x) + 2x^2\cos(1/x)(-x^{-2})$$
$$= 1 + 4x\sin(1/x) - 2\cos(1/x)$$

$x = 0$: $\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \left(1 + 2x\sin(1/x)\right) = 1$ by Squeeze Thm

So $g'(x)$ exists $\forall x \in \mathbb{R}$, i.e. g is diff. $\forall x \in \mathbb{R}$.

Now $g'(0) = 1$.

Want: $x_n, y_n \to 0$ s.t. $\cos\left(\frac{1}{x_n}\right) = 1$, $\cos\left(\frac{1}{y_n}\right) = -1$.

Let $x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{(2n+1)\pi}$

Then $g'(x_n) = 1 + 0 - 2 = -1$

$g'(y_n) = 1 + 0 + 2 = 3$

Since $x_n, y_n \to 0$, so $g'(x)$ takes on both +ve and -ve values in every nbhd of 0.

The last assertion follows immediately from Thm 6.2.7. //